

Conformal Ricci Collineations of Plane Symmetric Static Spacetimes

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Abstract

This article explores the Conformal Ricci Collineations (CRCs) for the plane-symmetric static spacetime. The non-linear coupled CRC equations are solved to get the general form of conformal Ricci symmetries. In the non-degenerate case, it turns out that the dimension of the Lie algebra of CRCs is finite. In the case where the Ricci tensor is degenerate, it found that the algebra of CRCs for the plane-symmetric static spacetime is mostly, but not always, infinite dimensional. In one case of degenerate Ricci tensor, we solved the differential constraints completely and a spacetime metric is obtained along with CRCs. We found ten possible cases of finite and infinite dimensional Lie algebras of CRCs for the considered spacetime.

Keywords: Symmetries of a spacetimes, Conformal Ricci Collineation, Plane Symmetric Static Spacetimes.

1 Introduction

The general theory of relativity proposed by Einstein is an elegant theory of gravitation which has provided an understanding of the mystery of gravity at least at the classical level. An important benefit of the general theory of relativity is that it helps us understand the large-scale structure of the universe. In theoretical physics one has two main tools to study the properties of evolution of dynamical systems: Symmetries of the equations

of motion and Collineations (symmetries) of the spacetime. The symmetries (symmetry analysis method) of differential equations (ordinary and partial) are a powerful method to find the exact (invariant) solutions of Einstein field equations [5, 6, 8, 22, 23, 27]. The symmetries of the spacetime play a central role in the classification of the exact solutions of the Einstein field equations. The classification of spacetimes according to different types of symmetries is an important part of recent research in the field of general relativity [18, 25]. The most basic symmetries of a spacetimes are the *Killing symmetries* or *isometries*. Killing symmetries of a spacetimes are vector fields along which the metric tensor remains invariant under the Lie transport. The components of Killing vector fields satisfy the *Killing equation* which is given as

$$g_{ij,k} X^k + g_{jk} X^k_{,i} + g_{ik} X^k_{,j} = 0, \quad (1)$$

where g_{ij} denotes the metric tensor components, X^k are the components of the Killing vector field and the commas in the subscript are used for partial derivatives with respect to spacetime coordinates. The homothetic symmetries are obtained by replacing the right-hand side of equation (1) by $2\alpha g_{ij}$, α is an arbitrary constant. Ricci collineation (RC) of a spacetime is achieved by replacing the metric tensor g_{ij} by the Ricci tensor R_{ij} . This type of collineation satisfies the equation [20]:

$$R_{ij,k} X^k + R_{jk} X^k_{,i} + R_{ik} X^k_{,j} = 0. \quad (2)$$

If we take $2\alpha R_{ij}$ instead of zero on the right-hand side of equation (2), we get

$$R_{ij,k} X^k + R_{jk} X^k_{,i} + R_{ik} X^k_{,j} = 2\alpha R_{ij}. \quad (3)$$

If α is an arbitrary constant in equation (3), the vector field X is known as *homothetic RC* or the *Ricci inheritance symmetries* [14, 15] and if α is an arbitrary function of the spacetime coordinates, then X becomes the *conformal Ricci Collineations* [26]. For Riemannian manifolds, CRC were called concircular vector fields.

The static spherical symmetric spacetimes according to the RC are classified and the relation between isometries and the RC was established in [7]. The RC for Bianchi type I, III and Kantowski-Sachs spacetimes have been found in [11]. The RC of Bainchi type II, VIII and IX spacetimes are presented by Yavuz and Camci [28]. A complete classification of cylindrically symmetric static Lorentzian manifold according to their RC is provided by Qadir et al [24]. They are also compared with Killing and homothetic symmetries. Camci and Barnes [12] obtained the RC and Ricci inheritance collineation in the non-degenerate case for FRW spacetimes. The relation between RC and isometries for static plane symmetric spacetimes is established in [17]. RC for maximally symmetric transverse spacetimes are considered for both degenerate and non-degenerate cases in [1, 2]. The Ricci inheritance symmetries of spherically symmetric spacetimes and Friedman models are discussed in [9] and [10], respectively.

The RC have been investigated for many spacetimes, however Ricci inheritance symmetries have been discussed for only few spacetimes. Recently, Hussain et al [19] and Ali and Khan [3] explored the Ricci inheritance symmetries in Bainchi type I spacetimes and plane symmetric static spacetimes, respectively. Conformal Killing vector fields (CKVFs) are symmetries of the metric tensor, while conformal Ricci collineations (CRC) are symmetries of Ricci tensor. In literature a relationship between the conformal factors of CKVFs and CRCs is established [13]. It is also well known that when the Ricci tensor is non-degenerate, the maximum dimension of the group of CRCs is 15 and this dimension is achieved only if the Ricci tensor when taken as a metric is conformally flat. Recently, a relationship between CRCs and CKVFs for pp-waves has been also discussed in detail [21]. Ali and Suhail [4] solved CKVFs equations along with the equation (3) but they replaced the Ricci tensor R_{ij} in the right hand side by the metric tensor g_{ij} . They solved it for static plane-symmetric four dimensional Lorentzian manifold and the vector field is called the concircular vector fields. Our aim of this work is to classify a plane-symmetric static spacetime according to the conformal Ricci symmetries with the help of the conformal Ricci collineation equation (3) by taking the conformal factor α to be a general function of the spacetime coordinates t, x, y and z . This article is organized as follows: In section 2, ten coupled conformal Ricci collineation equations are obtained. These equations are solved for non-degenerate and generate cases in sections 3 and 4, respectively and the conformal Ricci collineations are obtained. A brief summary of the work and some discussion on the obtained results are introduced in the last section.

2 Conformal Ricci collineation equations

We take the plane-symmetric static spacetime in the the convention coordinates ($x^0 = t, x^1 = x, x^2 = y, x^3 = z$), in the form

$$ds^2 = -e^{2\mu(x)} dt^2 + dx^2 + e^{2\nu(x)} (dy^2 + dz^2), \quad (4)$$

where μ and ν are functions of x only. The non-zero Ricci tensor components for the space time (4) are

$$\begin{cases} R_{00} = -e^{2\mu} (\mu'' + 2\mu' \nu' + \mu'^2) = A(x), \\ R_{11} = 2(\nu'' + \nu'^2) - (\mu'' + \mu'^2) = B(x), \\ R_{22} = R_{33} = e^{2\nu} (\nu'' + \mu' \nu' + 2\nu'^2) = C(x), \end{cases} \quad (5)$$

where the prime denotes the derivative with respect to x . Using Ricci components above in equation (3) yields the following ten partial differential equations:

$$A' X^1 + 2 A X^0_{,0} = 2\alpha A, \quad (6)$$

$$A X^0_{,1} + B X^1_{,0} = 0, \quad (7)$$

$$A X_{,2}^0 + C X_{,0}^2 = 0, \quad (8)$$

$$A X_{,3}^0 + C X_{,0}^3 = 0, \quad (9)$$

$$B' X^1 + 2 B X_{,1}^1 = 2 \alpha B, \quad (10)$$

$$B X_{,2}^1 + C X_{,1}^2 = 0, \quad (11)$$

$$B X_{,3}^1 + C X_{,1}^3 = 0, \quad (12)$$

$$C' X^1 + 2 C X_{,2}^2 = 2 \alpha C, \quad (13)$$

$$X_{,3}^2 + X_{,2}^3 = 0, \quad (14)$$

$$C' X^1 + 2 C X_{,3}^3 = 2 \alpha C. \quad (15)$$

3 CRCs for non-degenerate Ricci tensor

In this section, we solve Eqs. (6)–(15) by considering all the Ricci tensor components to be non-zero, that is $\det(R_{ij}) = A(x)B(x)C^2(x) \neq 0$. We shall use direct integration technique to find the components X^0 , X^1 , X^2 and X^3 of the conformal Ricci symmetries and the function α . The process of obtaining these components is explained as the following:

We differentiate the Eq. (8) with respect to z , Eq. (9) with respect to y and Eq. (14) with respect to t to obtain a relation

$$X_{,yz}^0 = X_{,tz}^2 = X_{,ty}^3 = 0. \quad (16)$$

Similarly, Eqs. (11), (12) and (14), after differentiating with respect to z , y , x , respectively and some simple algebraic calculation, gives

$$X_{,yz}^1 = X_{,xz}^2 = X_{,xy}^3 = 0. \quad (17)$$

By applicable a similar manor between Eqs. (8), (9), (11), (12), (13), (14) and (15), gets

$$X_{,yy}^i - X_{,zz}^i = X_{,yy}^j + X_{,zz}^j = 0, \quad i = 0, 1, \quad j = 2, 3. \quad (18)$$

Integrating Eqs. (16) and (17), substituting the results in (18), again integrate (18) and

solving the Eqs. (6), (8), (9), (11), (12) and (14), we get

$$\left\{ \begin{array}{l} X^0 = F^0 - 2 \sqrt{\frac{C(x)}{A(x)}} \left[y G^1 + z G^2 + (y^2 + z^2) G^3 \right]_{,t}, \\ X^1 = F^1 - \frac{C(x)}{B(x)} \left(\left[\frac{A(x)}{C(x)} \right]' \sqrt{\frac{C(x)}{A(x)}} \left[y G^1 + z G^2 + (y^2 + z^2) G^3 \right] \right. \\ \quad \left. + \left[y H^1 + z H^2 + (y^2 + z^2) H^3 \right]_{,x} \right), \\ X^2 = H^1 + 2 H^3 y + d_0 z + d_1 (y^2 - z^2) + 2 d_2 y z + 2 \sqrt{\frac{A(x)}{C(x)}} \left[G^1 + 2 G^3 y \right], \\ X^3 = H^2 - d_0 y + 2 H^3 z + d_2 (z^2 - y^2) + 2 d_1 y z + 2 \sqrt{\frac{A(x)}{C(x)}} \left[G^2 + 2 G^3 z \right], \end{array} \right. \quad (19)$$

and

$$\alpha = X_{,t}^0 + \frac{A'(x) X^1}{2 A(x)}, \quad (20)$$

where $F^i = F^i(t, x)$, $G^j = G^j(t)$ and $H^j = H^j(x)$ are functions of integration while d_k , are constants of integration for all $i = 0, 1$, $j = 1, 2, 3$ and $k = 0, 1, 2$. Substituting these values of X^0 , X^1 , X^2 and X^3 in the system (6)-(15), expanding it with the aid of *Mathematica Program* and set the coefficients involving y and z and various products equal zero, give to the following set of over-determined equations:

$$A F_{,x}^0 + B F_{,t}^1 = 0, \quad (21)$$

$$\left[\frac{B}{A} \right]' F^1 + 2 \left[\frac{B}{A} \right] (F_{,x}^1 - F_{,t}^0) = 0, \quad (22)$$

$$\left[\frac{A}{C} \right]' F^1 - 2 \left[\frac{A}{C} \right] \left[4 G^3 \sqrt{\frac{A}{C}} + 2 H^3 - F_{,t}^0 \right] = 0, \quad (23)$$

$$\left(\frac{A^2}{B C} \left[\frac{C}{A} \right]'^2 \right)' G^j + \left[\frac{C}{A} \right]' \left(\left[\frac{A}{C} \right]^{3/2} \left[\frac{C^2}{A B} \right]' H_{,x}^j \right. \quad (24)$$

$$\left. - 2 \left[\frac{\sqrt{A C}}{B} \right] H_{,xx}^j - 4 G_{,tt}^j \right) = 0, \quad j = 1, 2, 3,$$

$$\left[\frac{A}{C} \right]'^2 G^j + \left[\frac{A}{C} \right]' \sqrt{\frac{A}{C}} H_{,x}^j + \left[\frac{4 A B}{C^2} \right] \left[d_j \sqrt{\frac{A}{C}} + G_{,tt}^j \right] = 0, \quad j = 1, 2, 3, \quad (25)$$

where $d_3 = 0$. Differentiating the Eq. (25) with respect to t , the following equation is obtained:

$$\left[\frac{A}{C}\right]'^2 G_{,t}^j + \left[\frac{4AB}{C^2}\right] [G_{,tt}^j] = 0, \quad j = 1, 2, 3. \quad (26)$$

The above equation gives rise to the following two possibilities: (I): $\left[\frac{A(x)}{C(x)}\right]' \neq 0$; (II): $\left[\frac{A(x)}{C(x)}\right]' = 0$. We shall discuss each case in turn.

Case (I): In this case we consider $\left[\frac{A(x)}{C(x)}\right]' \neq 0$. Then Eq. (26) leads to

$$B(x) = \frac{C^2(x)}{4c_0 A(x)} \left[\frac{A(x)}{C(x)}\right]'^2, \quad (27)$$

where c_0 is an arbitrary constant. Now we study two subcases: The first one when c_0 positive and the second when c_0 negative as follows:

Case (I-A): Here, taking $c_0 = a_0^2 > 0$ and solving Eq. (26) we get:

$$G^j(t) = e_{j1} + e_{j2} \cos[a_0 t] + e_{j3} \sin[a_0 t], \quad j = 1, 2, 3, \quad (28)$$

where e_{ij} are constants of integration for all $i, j = 1, 2, 3$. Substituting the values of $B(x)$ and $G^j(t)$ in Eq. (24) and solving the resulting equation we obtain:

$$H^j(x) = f_j - \frac{d_j A(x)}{a_0^2 C(x)} - 2e_{j1} \sqrt{\frac{A(x)}{B(x)}}, \quad j = 1, 2, 3, \quad (29)$$

where f_j are constants of integration for all $j = 1, 2, 3$. From Eq. (23) we have

$$F^1(t, x) = \frac{A(x) C(x) (4f_3 - 2F_{,t}^0) - 8A(x) \sqrt{A(x) C(x)} (e_{32} \cos[a_0 t] + e_{33} \sin[a_0 t])}{A(x) C'(x) - C(x) A'(x)}. \quad (30)$$

and from Eq. (22) we get

$$F^0(t, x) = H^4(x) + G^4(t) \sqrt{\frac{C(x)}{A(x)}} + 2 \sqrt{\frac{A(x)}{C(x)}} \left[\left(\frac{e_{32}}{a_0} \right) \sin[a_0 t] - \left(\frac{e_{33}}{a_0} \right) \cos[a_0 t] \right], \quad (31)$$

where $G^4(t)$ and $H^4(x)$ are functions of integration. Now, Eq. (21) becomes:

$$\left[\frac{A(x)}{C(x)}\right]' [G_{,tt}^4(t) + a_0^2 G^4(t)] = 2a_0^2 \left[\frac{A(x)}{C(x)}\right]^{3/2} H_{,x}^4. \quad (32)$$

Solving the above equation we obtain

$$G^4(t) = e_{41} + e_{42} \cos[a_0 t] + e_{43} \sin[a_0 t], \quad (33)$$

$$H^4(x) = f_4 - e_{41} \sqrt{\frac{C(x)}{A(x)}}, \quad (34)$$

where f_4 and e_{4j} are constants of integration for all $j = 1, 2, 3$. Hence, we have the following components of CRCs:

$$\left\{ \begin{array}{l} X^0 = a_1 + \sqrt{\frac{C(x)}{A(x)}} \left[\left(a_2 + a_3 y + a_4 z + a_5 \left[a_0^2 (y^2 + z^2) + \frac{A(x)}{C(x)} \right] \right) \cos [a_0 t] \right. \\ \quad \left. + \left(a_6 + a_7 y + a_8 z + a_9 \left[a_0^2 (y^2 + z^2) + \frac{A(x)}{C(x)} \right] \right) \sin [a_0 t] \right], \\ X^1 = \frac{2 a_0 C(x) \sqrt{C(x)}}{\sqrt{A(x)} [A(x) C'(x) - C(x) A'(x)]} \left[a_{10} + a_{11} y + a_{12} z \right. \\ \quad + \left(a_6 + a_7 y + a_8 z + a_9 \left[a_0^2 (y^2 + z^2) - \frac{A(x)}{C(x)} \right] \right) \cos [a_0 t] \\ \quad \left. - \left(a_2 + a_3 y + a_4 z + a_5 \left[a_0^2 (y^2 + z^2) - \frac{A(x)}{C(x)} \right] \right) \sin [a_0 t] \right], \\ X^2 = \frac{a_0}{2} \left[a_{13} - 2 a_{10} y + a_{14} z + a_{11} \left(z^2 - y^2 + \frac{A(x)}{a_0^2 C(x)} \right) \right] \\ \quad + \sqrt{\frac{A(x)}{C(x)}} \left[\left(\frac{h_7}{a_0} + 2 a_0 a_9 y \right) \cos [a_0 t] - \left(\frac{a_3}{a_0} + 2 a_0 a_5 y \right) \sin [a_0 t] \right], \\ X^3 = \frac{a_0}{2} \left[a_{15} - a_{14} y - 2 a_{10} z + a_{12} \left(y^2 - z^2 + \frac{A(x)}{a_0^2 C(x)} \right) \right] \\ \quad + \sqrt{\frac{A(x)}{C(x)}} \left[\left(\frac{h_8}{a_0} + 2 a_0 a_9 z \right) \cos [a_0 t] - \left(\frac{h_4}{a_0} + 2 a_0 a_5 z \right) \sin [a_0 t] \right], \end{array} \right. \quad (35)$$

and

$$\begin{aligned}
\alpha = & \frac{a_0 \sqrt{A(x)C(x)}}{A(x)C'(x) - C(x)A'(x)} \left[(a_{10} + a_{11}y + a_{12}z) \left(\frac{\sqrt{C(x)}A'(x)}{\sqrt{A(x)}} \right) \right. \\
& + \left([a_6 + a_7y + a_8z] C'(x) + a_9 [a_0^2(y^2 + z^2) C'(x) \right. \\
& \left. \left. - 2A'(x) + \frac{A(x)C'(x)}{C(x)} \right] \right) \cos[a_0 t] - \left([a_2 + a_3y + a_4z] C'(x) \right. \\
& \left. \left. + a_5 \left[a_0^2(y^2 + z^2) C'(x) - 2A'(x) + \frac{A(x)C'(x)}{C(x)} \right] \right) \sin[a_0 t] \right], \tag{36}
\end{aligned}$$

where $B(x) = \frac{A(x)}{4a_0^2} \left(\frac{A'(x)}{A(x)} - \frac{C'(x)}{C(x)} \right)^2$ and $a_i, i = 0, 1, \dots, 15$ are arbitrary constants of integration.

Case (I-B): In this case we take $c_0 = -a_0^2 < 0$. By a similar method, we obtain the

following components of CRCs:

$$\left\{ \begin{array}{l}
X^0 = a_1 + \sqrt{\frac{C(x)}{A(x)}} \left[\left(a_2 + a_3 y + a_4 z - a_5 \left[a_0^2 (y^2 + z^2) - \frac{A(x)}{C(x)} \right] \right) \cosh [a_0 t] \right. \\
\qquad \qquad \qquad \left. + \left(a_6 + a_7 y + a_8 z - a_9 \left[a_0^2 (y^2 + z^2) - \frac{A(x)}{C(x)} \right] \right) \sinh [a_0 t] \right], \\
X^1 = \frac{2 a_0 C(x) \sqrt{C(x)}}{\sqrt{A(x)} [A(x) C'(x) - C(x) A'(x)]} \left[a_{10} + a_{11} y + a_{12} z \right. \\
\qquad \qquad \qquad + \left(a_6 + a_7 y + a_8 z - a_9 \left[a_0^2 (y^2 + z^2) + \frac{A(x)}{C(x)} \right] \right) \cosh [a_0 t] \\
\qquad \qquad \qquad \left. - \left(a_2 + a_3 y + a_4 z - a_5 \left[a_0^2 (y^2 + z^2) + \frac{A(x)}{C(x)} \right] \right) \sinh [a_0 t] \right], \\
X^2 = \frac{a_0}{2} \left[a_{13} - 2 a_{10} y + a_{14} z + a_{11} \left(z^2 - y^2 - \frac{A(x)}{a_0^2 C(x)} \right) \right] \\
\qquad \qquad \qquad - \sqrt{\frac{A(x)}{C(x)}} \left[\left(\frac{h_7}{a_0} - 2 a_0 a_9 y \right) \cosh [a_0 t] - \left(\frac{a_3}{a_0} - 2 a_0 a_5 y \right) \sinh [a_0 t] \right], \\
X^3 = \frac{a_0}{2} \left[a_{15} - a_{14} y - 2 a_{10} z + a_{12} \left(y^2 - z^2 - \frac{A(x)}{a_0^2 C(x)} \right) \right] \\
\qquad \qquad \qquad - \sqrt{\frac{A(x)}{C(x)}} \left[\left(\frac{h_8}{a_0} - 2 a_0 a_9 z \right) \cosh [a_0 t] - \left(\frac{h_4}{a_0} - 2 a_0 a_5 z \right) \sinh [a_0 t] \right],
\end{array} \right. \quad (37)$$

and

$$\begin{aligned}
\alpha = & \frac{a_0 \sqrt{A(x) C(x)}}{A(x) C'(x) - C(x) A'(x)} \left[(a_{10} + a_{11} y + a_{12} z) \left(\frac{\sqrt{C(x)} A'(x)}{\sqrt{A(x)}} \right) \right. \\
& + \left([a_6 + a_7 y + a_8 z] C'(x) - a_9 [a_0^2 (y^2 + z^2) C'(x) \right. \\
& \left. \left. + 2 A'(x) + \frac{A(x) C'(x)}{C(x)} \right] \right) \cosh [a_0 t] - \left([a_2 + a_3 y + a_4 z] C'(x) \right. \\
& \left. \left. - a_5 \left[a_0^2 (y^2 + z^2) C'(x) + 2 A'(x) + \frac{A(x) C'(x)}{C(x)} \right] \right) \sinh [a_0 t] \right], \tag{38}
\end{aligned}$$

where $B(x) = -\frac{A(x)}{4a_0^2} \left(\frac{A'(x)}{A(x)} - \frac{C'(x)}{C(x)} \right)^2$ and $a_i, i = 0, 1, \dots, 15$ are arbitrary constants of integration.

Case (II): In this case we consider $\left[\frac{A(x)}{C(x)} \right]' = 0$. Then the components of CRCs are given by:

$$\left\{ \begin{aligned}
X^0 &= a_1 + a_2 t + a_3 y + a_4 z + a_5 \left[a_0 t^2 - y^2 - z^2 - \left(\int \sqrt{\frac{B(x)}{C(x)}} dx \right)^2 \right] \\
&\quad + 2 a_6 t y + 2 a_7 t z + (a_8 + 2 a_9 t) \left(\int \sqrt{\frac{B(x)}{C(x)}} dx \right), \\
X^1 &= \sqrt{\frac{C(x)}{B(x)}} \left[a_{10} - a_0 a_8 t + a_{11} y + a_{12} z \right. \\
&\quad \left. - a_9 \left[a_0 t^2 + y^2 + z^2 - \left(\int \sqrt{\frac{B(x)}{C(x)}} dx \right)^2 \right] \right. \\
&\quad \left. + (a_2 + 2 a_0 a_5 t + 2 a_6 y + 2 a_7 z) \left(\int \sqrt{\frac{B(x)}{C(x)}} dx \right) \right], \\
X^2 &= a_{13} - a_0 a_3 t + a_2 y + a_{14} z - a_6 \left[a_0 t^2 + y^2 + z^2 - \left(\int \sqrt{\frac{B(x)}{C(x)}} dx \right)^2 \right] \\
&\quad + 2 a_0 a_5 t y + 2 a_7 y z - (a_{11} - 2 a_9 y) \left(\int \sqrt{\frac{B(x)}{C(x)}} dx \right), \\
X^3 &= a_{15} - a_0 a_4 t - a_{14} y + a_2 z - a_7 \left[a_0 t^2 + y^2 + z^2 + \left(\int \sqrt{\frac{B(x)}{C(x)}} dx \right)^2 \right] \\
&\quad + 2 a_0 a_5 t z + 2 a_6 y z - (a_{12} - 2 a_9 z) \left(\int \sqrt{\frac{B(x)}{C(x)}} dx \right),
\end{aligned} \right. \tag{39}$$

and

$$\begin{aligned}
\alpha = & (a_2 + 2a_0 a_5 t + 2a_6 y + 2a_7 z) \left[1 + \left(\frac{C'(x)}{2\sqrt{B(x)C(x)}} \right) \int \sqrt{\frac{B(x)}{C(x)}} dx \right] \\
& + \frac{(a_{10} - a_0 a_8 t + a_{11} y + a_{12} z) C'(x)}{2\sqrt{B(x)C(x)}}, \\
& + h_9 \left(2 \int \sqrt{\frac{B(x)}{C(x)}} dx - \frac{C'(x)}{2\sqrt{B(x)C(x)}} \left[a_0 t^2 + y^2 + z^2 - \left(\int \sqrt{\frac{B(x)}{C(x)}} dx \right)^2 \right] \right),
\end{aligned} \tag{40}$$

where $A(x) = a_0 C(x)$ and $a_i, i = 0, 1, \dots, 15$ are arbitrary constants.

4 CRCs for degenerate Ricci tensor

In this case, the Ricci tensor R_{ij} is degenerate, that is, $\det(R_{ij}) = 0$, i.e., $A(x)B(x)C(x) = 0$, then there are six possible cases depending on whether one or two components of the Ricci tensor are zero. The conformal Ricci collineations are given in each of these cases; however we omit the basic calculation and give the final form of CRCs in each case.

Case (IV): In this case $A(x) = B(x) = 0$ and $C(x) \neq 0$. We are left with the following seven equations:

$$C' X^1 + 2C X_y^2 = 2\alpha C, \tag{41}$$

$$C' X^1 + 2C X_z^3 = 2\alpha C, \tag{42}$$

$$X_t^2 = X_x^2 = X_t^3 = X_x^3 = 0, \tag{43}$$

$$X_z^2 + X_y^3 = 0. \tag{44}$$

Eqs. (43) and (44) give $X^2 = F_y(y, z)$ and $X^3 = G(z) - F_z(y, x)$. Subtracting Eqs. (41) and (42) and differentiating the resulting equation with respect to y , we get an equation $F_{yyy} + F_{yzz} = 0$. Now, we have the general solution of conformal Ricci collineations as the following:

$$\begin{cases} X^0 = X^0(t, x, y, z), & X^1 = X^1(t, x, y, z), \\ X^2 = \Psi_{,y}(y, z), & X^3 = -\Psi_{,z}(y, z), \end{cases} \tag{45}$$

and $\alpha = \Psi_{,yy} + \frac{C'(x)X^1}{2C(x)}$, where $X^0(t, x, y, z)$, $X^1(t, x, y, z)$ and $C(x)$ are arbitrary functions while $\Psi(y, z)$ is a function satisfy the relation $\Psi_{,yy} + \Psi_{,zz} = 0$.

Case (V): In this case we take $A(x) = C(x) = 0$ and $B(x) \neq 0$. Here we are left with the following four equations:

$$B' X^1 + 2 C X_x^1 = 2 \alpha B, \quad (46)$$

$$X_{,i}^1 = 0, \text{ where } i = 0, 2, 3. \quad (47)$$

Solving these four equations we observe the three (one temporal and two spatial) conformal Ricci collineations are arbitrary functions of the spacetime coordinates (t, x, y, z) and the spatial CRC X^1 is given as

$$X^1 = \frac{a_1 + \int \alpha(x) \sqrt{B(x)} dx}{\sqrt{B(x)}}, \quad (48)$$

and the conformal factor $\alpha = \alpha(x)$ is a function of x only, where a_1 is an arbitrary constant. In this case the differential constraints are solved completely and the metric which admit the above CRC is also obtained as follows:

$$ds^2 = -a_0^2 (x_0 + x)^{2\mu_0} dt^2 + dx^2 + c_0^2 (x_0 + x)^{1-\mu_0} (dy^2 + dz^2), \quad (49)$$

where a_0 , c_0 , x_0 and μ_0 are arbitrary constants. Therefore, the Ricci tensor form can be written as:

$$ds_{Ric}^2 = A(x) dt^2 + B(x) dx^2 + C(x) (dy^2 + dz^2) = \left[\frac{(1 - \mu_0)(1 + 3\mu_0)}{2(x + x_0)^2} \right] dx^2. \quad (50)$$

The above Ricci metric admits conformal Ricci vector field in the form

$$X = X^0(t, x, y, z) \frac{\partial}{\partial t} + (x + x_0) \left[\alpha_0 + \int \frac{\alpha(x)}{x + x_0} dx \right] \frac{\partial}{\partial x} + \sum_{i=2}^3 X^i(t, x, y, z) \frac{\partial}{\partial x^i}, \quad (51)$$

where α_0 is an arbitrary constant while X^0 , X^2 and X^3 are arbitrary functions of the coordinates (t, x, y, z) .

Case (VI): In this case we consider $B(x) = C(x) = 0$ and $A(x) \neq 0$. Here we are left with the following equations to be dealt with:

$$A' X^1 + 2 A X_t^0 = 2 \alpha A, \quad (52)$$

$$X_{,i}^0 = 0, \text{ where } i = 1, 2, 3. \quad (53)$$

Solving the above equations we get the following CRCs.

$$X^0 = \Psi(t), \quad X^i = X^i(t, x, y, z), \text{ for } i = 1, 2, 3, \quad (54)$$

and $\alpha = \Psi'(t) + \frac{A'(x) X^1}{2 A(x)}$ where $\Psi(t)$, $X^1(t, x, y, z)$, $X^2(t, x, y, z)$, $X^3(t, x, y, z)$ and $A(x)$ are arbitrary functions.

Case (VII): In this case $A(x) \neq 0$, $B(x) \neq 0$ and $C(x) = 0$. Then the Eqs. (6)–(15) reduced as follows:

$$A' X^1 + 2 A X_t^0 = 2 \alpha A, \quad (55)$$

$$B' X^1 + 2 B X_x^1 = 2 \alpha B, \quad (56)$$

$$A X_x^0 + B X_t^1 = 0, \quad (57)$$

$$X_y^0 = X_z^0 = X_y^1 = x_z^1 = 0, \quad (58)$$

Solving the Eqs. (55)–(58), we get the components of CRCs as the following:

$$\begin{cases} X^0 = G(t) - \int \frac{B(x)}{A(x)} X_{,t}^1(t, x) dx, & X^1 = X^1(t, x), \\ X^2 = X^2(t, x, y, z), & X^3 = X^3(t, x, y, z), \end{cases} \quad (59)$$

and $\alpha = X_{,x}^1 + \frac{B'(x) X^1}{2 B(x)}$ where $G(t)$, $A(x)$, $B(x)$, $X^2(t, x, y, z)$ and $X^3(t, x, y, z)$ are arbitrary functions while $X^1(t, x)$ satisfy the following partial integral differential equation:

$$\left(\frac{A'(x)}{A(x)} - \frac{B'(x)}{B(x)} \right) X^1 - x_{,x}^1 = G(t) + \int \frac{B(x)}{A(x)} X_{,tt}^1 dx. \quad (60)$$

Case (VIII): In this case $A(x) \neq 0$, $C(x) \neq 0$ and $B(x) = 0$. Then the CRC equations are

$$A' X^1 + 2 A X_t^0 = 2 \alpha A, \quad (61)$$

$$C' X^1 + 2 C X_y^2 = 2 \alpha C, \quad (62)$$

$$C' X^1 + 2 C X_z^3 = 2 \alpha C, \quad (63)$$

$$A X_y^0 + C X_t^2 = 0, \quad (64)$$

$$A X_z^0 + C X_t^3 = 0, \quad (65)$$

$$X_z^2 + X_y^3 = 0, \quad (66)$$

$$X_x^i = 0, \quad i = 0, 2, 3. \quad (67)$$

We omit to write the basic steps involved in the solution of the above equations. After some mathematical calculations we obtain the following two solutions of the components of CRCs:

(VIII-1):

$$\begin{cases} X^0 = \Psi(t) - y \Phi(t) - z \Omega(t), & X^1 = X^1(t, x, y, z), \\ X^2 = a_0 \Phi(t) + F_{,y}(y, z), & X^3 = a_0 \Omega(t) - F_{,z}(y, z), \end{cases} \quad (68)$$

and $\alpha = \frac{C' X^1}{2C} + F_{,yy}$ such that $A(x) = a_0 C(x)$ and the function $F(y, z)$ satisfy $F_{,yy}(y, z) + F_{,zz}(y, z) = 0$ where $C(x)$, $X^1(t, x, y, z)$, $\Phi(t)$ and $\Omega(t)$ are arbitrary functions while a_0 is an arbitrary constant.

(VIII-2):

$$X^0 = \Psi(t), \quad X^1 = X^1(t, x, y, z), \quad X^2 = F_{,y}(y, z), \quad X^3 = -F_{,z}(y, z), \quad (69)$$

and $\alpha = \frac{C' X^1}{2C} + F_{,yy}$ where $X^1(t, x, y, z)$, $A(x)$, $C(x)$, $\Phi(t)$ and $\Omega(t)$ are arbitrary functions while the function $F(y, z)$ satisfy the relation $F_{,yy}(y, z) + F_{,zz}(y, z) = 0$.

Case (IX): In this case we take $B(x) \neq 0$, $C(x) \neq 0$ and $A(x) = 0$. This case gives the following CRCs:

$$\left\{ \begin{array}{l} X^0 = X^0(t, x, y, z), \\ X^1 = \sqrt{\frac{C(x)}{B(x)}} \left[a_1 + a_2 y + a_3 z + a_4 \left(y^2 + z^2 - \left[\int \sqrt{\frac{B(x)}{C(x)}} dx \right]^2 \right) \right. \\ \qquad \qquad \qquad \left. + 2(a_5 + a_6 y + a_7 z) \int \sqrt{\frac{B(x)}{C(x)}} dx \right], \\ X^2 = a_8 + 2a_5 y + a_9 z + a_6 \left(y^2 - z^2 - \left[\int \sqrt{\frac{B(x)}{C(x)}} dx \right]^2 \right) \\ \qquad \qquad \qquad + 2a_7 y z - (a_2 + 2a_4 y) \int \sqrt{\frac{B(x)}{C(x)}} dx, \\ X^3 = a_{10} - a_9 y + 2a_5 z + a_7 \left(z^2 - y^2 - \left[\int \sqrt{\frac{B(x)}{C(x)}} dx \right]^2 \right) \\ \qquad \qquad \qquad + 2a_6 y z - (a_3 + 2a_4 z) \int \sqrt{\frac{B(x)}{C(x)}} dx, \end{array} \right. \quad (70)$$

and

$$\begin{aligned} \alpha = & \frac{(a_1 + a_2 y + a_3 z) C'(x)}{2 \sqrt{B(x) C(x)}} \\ & + a_4 \left[\frac{C'(x)}{2 \sqrt{B(x) C(x)}} \left(y^2 + z^2 - \left[\int \sqrt{\frac{B(x)}{C(x)}} dx \right]^2 \right) - 2 \int \sqrt{\frac{B(x)}{C(x)}} dx \right], \quad (71) \\ & + (a_5 + a_6 y + a_7 z) \left[2 + \frac{C'(x)}{2 \sqrt{B(x) C(x)}} \int \sqrt{\frac{B(x)}{C(x)}} dx \right] \end{aligned}$$

where $B(x)$ and $C(x)$ are arbitrary functions, a_i , $i = 1, \dots, 10$ are arbitrary constants.

5 Conclusion

In this paper a plane symmetric static spacetime is classified according to its conformal Ricci collineations by taking the conformal factor α to be a general function of the space-time coordinates. The explicit forms of conformal Ricci vectors along with constraints on the components of the Ricci tensor are obtained. When the Ricci tensor is non-degenerate, we have obtained finite number of CRCs while the Ricci tensor is degenerate, it is concluded that the Lie algebras of the CRCs is not always finite. It is worth note that, the numbers of all possible cases of finite and infinite dimensional Lie algebras of CRCs for the plane symmetric static spacetime are *ten*. In the non-degenerate case there are *three* CRCs. Since we obtained highly non-linear differential constraints on the components of Ricci tensor, so we have not been able to get exact form of the metric except in case (V) in sect. 4, where we obtained the spacetime metric which admit infinite dimensional CRC.

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